CUHK Quant Trading Society

QUANT REPORT

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Why Trading Options in Efficient Markets Reduces Sharpe Ratio?

Introduction

Options have often been described as financial tools to help reduce risk. Yet, the use of such instruments could have detrimental effects on the risk-adjusted returns of a portfolio. We will investigate the practice of purchasing or writing options instead of implementing a buy-and-hold strategy. We will show that such acts would reduce Sharpe Ratios, assuming that stock prices are random.

In Section I, we will analytically show that the Sharpe Ratio of any arbitrary trading strategy is strictly lower than that of buy-and-hold, under the assumption that stock prices follow a random walk. In Section II, we will illustrate the above theory with a simulation. In Sections III and IV, we will show how this theory can be applied to options and why the use of options may reduce risk-adjusted returns. In Section V, we will conduct a similar analysis under the assumption that stock prices are not entirely random, i.e., if markets exhibit higher degrees of inefficiency. In Section VI, we will acknowledge the limitations of our research and suggest directions for further investigation.

The complete source code in this paper can be found in our GitHub repo (https://github.com/CUQTS).

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I. Analytical Proof

In this Section, we will analytically show that the Sharpe Ratio of any arbitrary trading strategy is strictly lower than that of buy-and-hold, under the assumption that stock prices follow a random walk.

Suppose there are only two investment securities, stock A and a risk-free asset B. Stock A has an expected return of r_A and volatility of σ_A , while the risk-free asset has an expected return of r_f. Suppose the return of stock A is completely random.

Moreover, suppose a trader follows an arbitrary long-only trading strategy. At time t, the strategy will produce a trading signal h using all information available up to time t. Trading signals are produced at every specified frequency (such as on every daily closing price or every tick). At time t, the trader will allocate h% of the portfolio to stock A, and (100-h)% to risk-free asset B.

Let us denote the portfolio return by r_p. In the following calculation, we will denote random variables by capitalizing the letters.

The Sharpe Ratio is given by $\frac{E[R_p]-r_f}{Var(R_p)}$. $R_p = HR_A + (1-H)r_f$. Note that H and R_A are independent since R_A is random and unpredictable by definition. Then the Sharpe Ratio of the trading strategy is as follows (see Proof (1) in the appendix).

$$Sharpe\ Ratio = \frac{E[R_A] - r_f}{\sqrt{\frac{E[H^2]}{E[H]^2}\left(E[R_A^2] - 2E[R_A]r_f + r_f^2\right) - \left(E[R_A]^2 - 2E[R_A]r_f + r_f^2\right)}}$$

Equation (1)

Suppose there is another portfolio that buys and holds stock A. This portfolio holds any combination of stock A and risk-free asset B without adjusting the portfolio in the future. The portfolio's Sharpe Ratio is given by

$$Sharpe\ Ratio = \frac{E[R_A] - r_f}{\sqrt{(E[R_A^2] - E[R_A]^2}}$$

Equation (2)

If we compare Equation (1) to Equation (2), we see that whether (1) is smaller than (2) depends entirely on Var(H), since $\mathrm{Var(H)} = E[H^2] - E[H]^2$. (1) is smaller than (2) if $\mathrm{Var(H)} > 0$, and (1) is equal to (2) if $\mathrm{Var(H)} = 0$.

This concludes our proof. Any arbitrary trading strategy should not produce constant values of h, or else, it is just a buy-and-hold strategy. Therefore, for any trading strategy, $\mathrm{Var}(\mathrm{H}) > 0$, and the Sharpe Ratio of the strategy is lower than that of a buy-and-hold strategy.

Of course, the above proof assumes that stock prices are completely random, i.e., markets are perfectly efficient. It also assumes that r_A, σ_A , and r_f are constants, which is almost certainly not true in real financial markets. The proof simply illustrates that traders cannot beat buy-and-hold strategies if underlying stock prices are random. It is, in fact, quite a simple proposition if we put it in simple words: If stock returns are inherently unpredictable, then it would be impossible to beat the market in terms of risk-adjusted returns. While this is a simple fact, there are important applications in derivatives, which we will discuss further in Section III.

II. Simulation of an arbitrary trading strategy

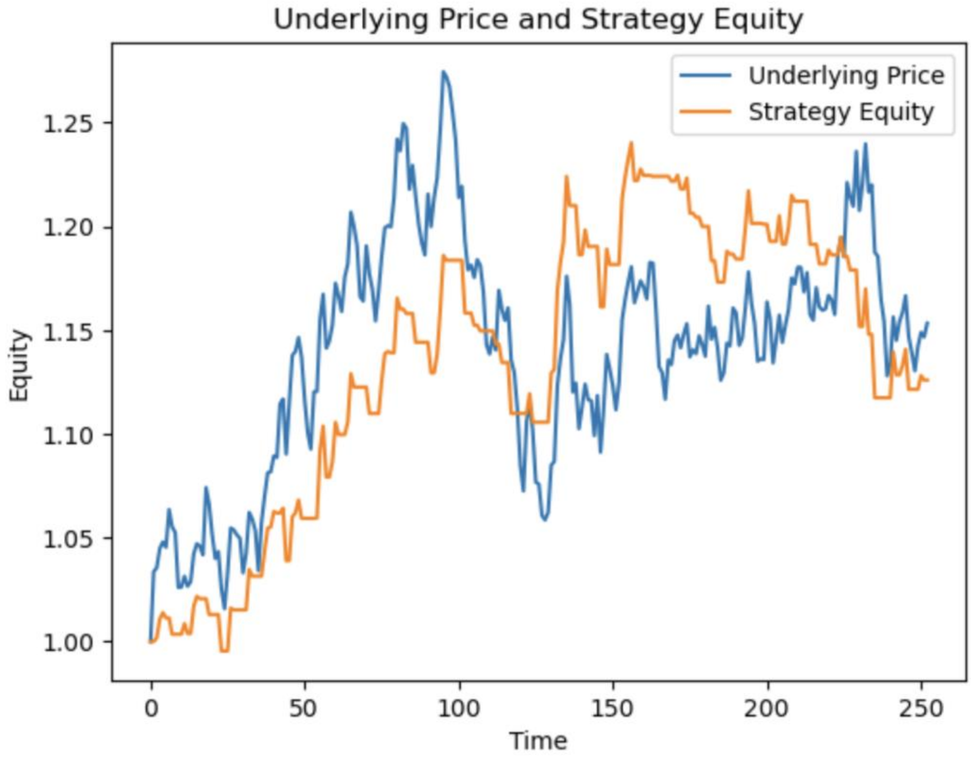
In this Section, we will illustrate the theory in Section I with a simulation.

Suppose there are a series of daily returns of stock X, $R_1, R_2. R_3, \dots, R_T$. R is an i.i.d. random variable that follows a log-normal distribution. Here we take the value 252 for T, approximately the number of trading days per year. Further, suppose that the risk-free rate is zero. R would follow the p.d.f. below.

$$f(R = x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(lnx-\mu)^2}{2\sigma^2}}$$

We arbitrarily choose $\mu = 0.0003$ and $\sigma = 0.012$. This should give us an expected annual return of 9.83% and a Sharpe Ratio of 0.492, which is typical of a stock index, assuming 252 trading days.

We implement a simple trading strategy: allocate 100% to stock X if the previous day's return is positive and allocate 100% to cash if the previous day's return is negative. The theoretical Sharpe Ratio calculated using Equation (1) in Section II is 0.35 (see Proof (2) in the appendix). We will run simulations in Python to validate.



(Figure 1)

Figure 1 illustrates an example of the trading strategy in action. We run similar simulations 100,000 times. The Sharpe Ratio of buy-and-hold is 0.492, while the Sharpe Ratio of the above strategy is 0.349. The Sharpe Ratio of the strategy is close to its theoretical Sharpe Ratio of 0.35. This validates the theory that an arbitrary trading strategy will produce sub-optimal risk-adjusted returns.

III. Options and their theoretical Sharpe Ratio

We can now extend our discussion to derivatives. Purchasing options is equivalent to implementing a trading strategy that has delta * 100% of the portfolio invested in the underlying and the rest in a risk-free asset. Note that an option's delta is dynamic, therefore there will be further buying and selling of the underlying stock in the future.

The theoretical Sharpe Ratio of a portfolio that buys call options can be calculated with the following formula.

$$Sharpe\ Ratio = \frac{E[R_A] - r_f}{\sqrt{\frac{E[H^2]}{E[H]^2}\left(E[R_A^2] - 2E[R_A]r_f + r_f^2\right) - \left(E[R_A]^2 - 2E[R_A]r_f + r_f^2\right)}}$$

$$E[H] = \frac{1}{T}\int_0^T\int_0^\infty P(X = x \mid t)N(d_1(x,t))\,dx\,dt$$

$$E[H^2] = \frac{1}{T}\int_0^T\int_0^\infty P(X = x \mid t)N\big(d_1(x,t)\big)^2\,dx\,dt$$

$$P(X = x \mid t) = \frac{1}{x\sigma\sqrt{t}\sqrt{2\pi}}e^{-\frac{(lnx-\mu t)^2}{2\sigma^2 t}}$$

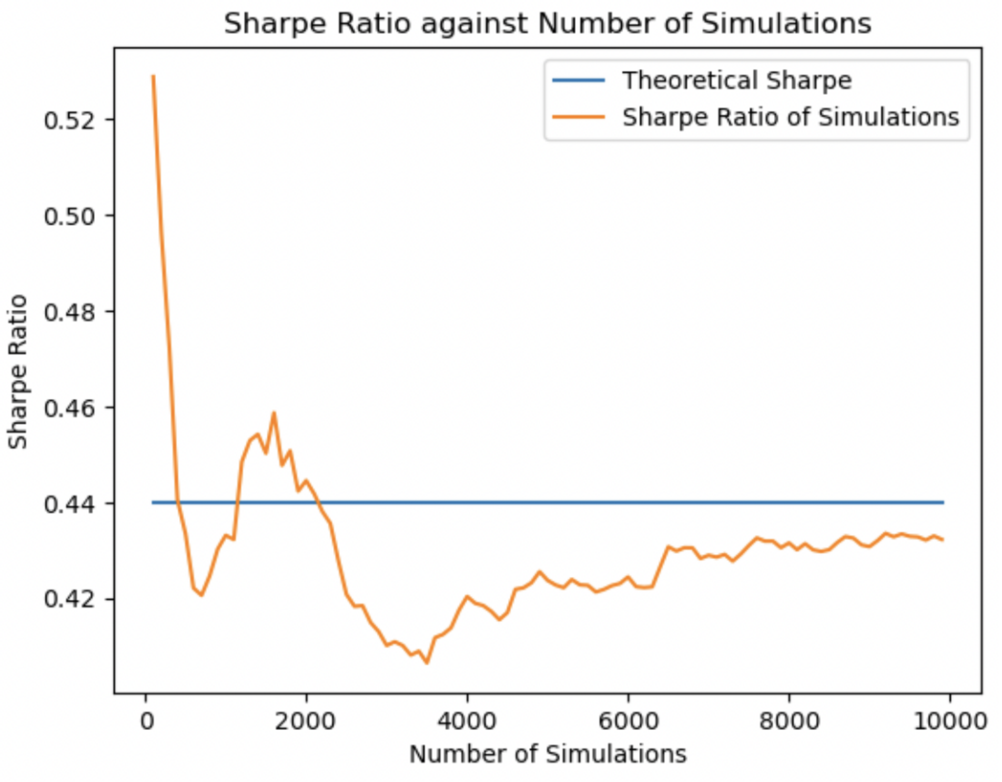
$$d_1(x,t) = \frac{ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

N is the cumulative standard normal distribution. K, r, σ, T are the strike price, risk-free-rate, annual volatility, and time to maturity, respectively. μ is the expected annual return of the underlying.

We use the same and figures as in Section II. Also, assume the risk-free-rate is zero. Then, the theoretical Sharpe Ratio of an ATM call option with one year to maturity will be 0.44 as calculated by the formula above. Note that the above Sharpe Ratio is lower than that of a buy-and-hold portfolio, which is 0.492, as mentioned in Section II.

IV. Simulation of a call option

We run 10,000 simulations and compute the Sharpe Ratio. We simulate a portfolio that continuously buys one year ATM call options after the previous option expired. The Sharpe Ratio of such a portfolio is 0.43, similar to our theoretical calculation. Please refer to GitHub for the complete source code.



(Figure 2)

V. Exceptions

So far, our discussion is based on the assumption that markets are perfectly efficient. In this Section, we will investigate the impact of options on the Sharpe Ratio if markets are less efficient.

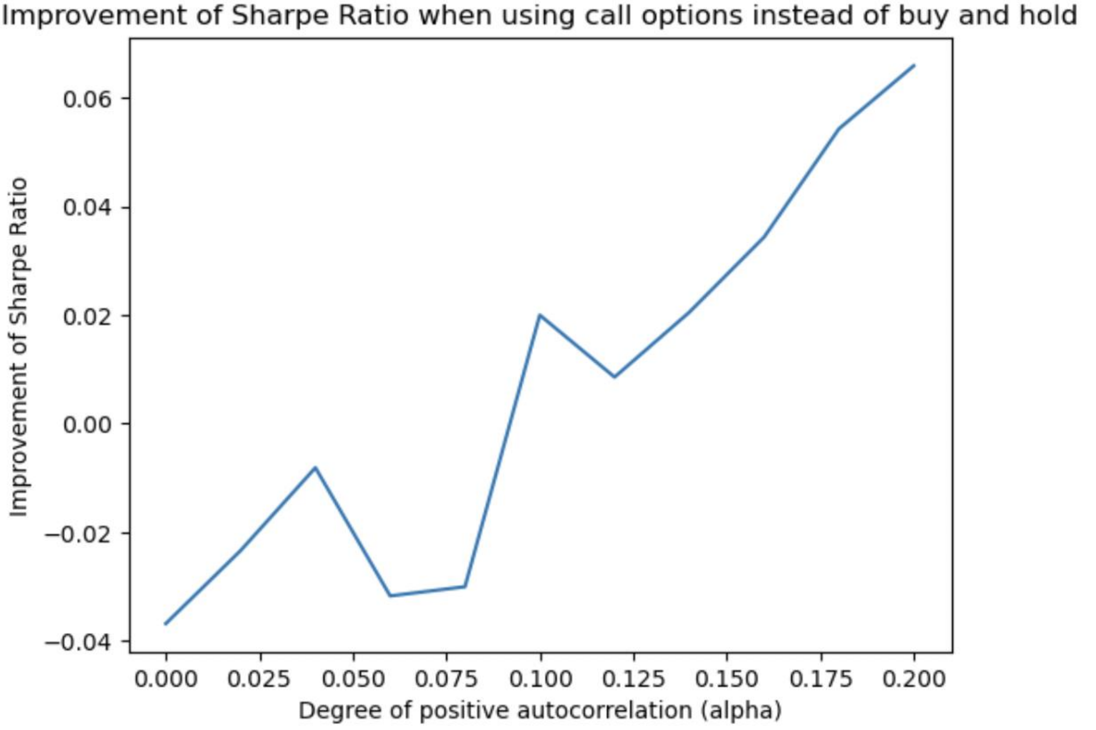
Suppose there are a series of daily returns of stock X, $R_1, R_2. R_3, \dots, R_T$. Contrary to Section III, R here exhibits positive autocorrelation. In mathematical terms,

$$f(R_{t+1} = x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(lnx-(\mu+\alpha lnR_t)^2}{2\sigma^2}}, \alpha > 0$$

Where α is the degree of positive autocorrelation. When α increases, the daily returns exhibit a greater degree of autocorrelation, implying that markets are less efficient.

Since R exhibits positive autocorrelation, the price of stock X will possess trending properties. We will use call options to capture the anomaly. Call options can be thought of as a trend-following strategy. This is because call options have positive gamma and a strictly increasing delta with respect to the underlying stock price. It can be thought of as buying more when the stock goes up and selling when the stock goes down, which is the defining characteristic of a trend-following strategy.

We will conduct 1,000 simulations for each different value of α and compare the Sharpe Ratio of purchasing a call option versus that of a buy-and-hold strategy.



(Figure 3)

As expected, the higher the degree of autocorrelation, the greater the improvement of the Sharpe Ratio. When markets are efficient, options reduce Sharpe Ratio. When markets exhibit strong trending behaviour, options improve Sharpe Ratio. According to the simulation in Figure 3, there is a gain of Sharpe Ratio using options when the value of exceeds around 0.1.

VI. Concluding thoughts

In this paper, we proved that assuming efficient markets, options, or any trading strategy, will reduce the Sharpe Ratio of portfolios compared to buy-and-hold. Of course, there are many limitations to our research. We list a few below.

1. Markets are not perfectly efficient. Prices do not follow geometric Brownian motions in real financial markets.
2. Sharpe Ratio is only one of the many performance metrics. A drop in Sharpe Ratio does not necessarily imply that the performance is worsened in all aspects.
3. Even if options reduce Sharpe Ratios, they still have many attractive characteristics as a derivative. For example, buying call options limits the maximum loss.

On a side note, in Section II, we showed that $\frac{E[H^2]}{E[H]^2}$ is indicative of Sharpe Ratio. Suppose two trading strategies are tested and they have identical Sharpe Ratios. The strategy with a greater $\frac{E[H^2]}{E[H]^2}$ value or greater variance of trading signals might imply that it is more capable of harvesting market anomalies. More research could be done to investigate the possibility of using $\frac{E[H^2]}{E[H]^2}$ or $Var(H)$ as a performance metric.

The complete source code in this paper can be found in our GitHub repo (https://github.com/CUQTS). Please contact us at cuqts.adm@gmail.com for any enquires.

VII. Appendix

Proof (1)

$$E[R_p] = E[HR_A + (1-H)r_f] = E[H]E[R_A] + (1-E[H])r_f$$

$$Var(R_p) = Var[HR_A + (1-H)r_f] = E[\left(HR_A + (1-H)r_f\right)^2] - \left(E\left[HR_A + (1-H)r_f\right]\right)^2$$
$$= E[H^2](E[R_A^2] - 2E[R_A]r_f + r_f^2) - E[H]^2(E[R_A]^2 - 2E[R_A]r_f + r_f^2)$$

$$Sharpe\ Ratio = \frac{E[R_p] - r_f}{\sqrt{Var(R_p)}} = \frac{E[H]E[R_A] - E[H]r_f}{\sqrt{E[H^2]\left(E[R_A^2] - 2E[R_A]r_f + r_f^2\right) - E[H]^2\left(E[R_A]^2 - 2E[R_A]r_f + r_f^2\right)}}$$
$$= \frac{E[R_A] - r_f}{\sqrt{\frac{E[H^2]}{E[H]^2}\left(E[R_A^2] - 2E[R_A]r_f + r_f^2\right) - \left(E[R_A]^2 - 2E[R_A]r_f + r_f^2\right)}}, \text{assuming E[H]} > 0 \text{ for a long strategy}$$

Proof (2)

$$E[R_A] = \int_0^\infty (x-1)\frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(lnx-\mu)^2}{2\sigma^2}} \approx 0.0003721$$

$$E[R_A^2] = \int_0^\infty (x-1)^2\frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(lnx-\mu)^2}{2\sigma^2}} \approx 0.0001443$$

$$E[H^2] = \int_1^\infty \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(lnx-\mu)^2}{2\sigma^2}} \approx 0.50997$$

$$E[H]^2 = \left(\int_1^\infty \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(lnx-\mu)^2}{2\sigma^2}}\right)^2 \approx 0.26007$$

$$Sharpe\ Ratio = \frac{E[R_A] - r_f}{\sqrt{\frac{E[H^2]}{E[H]^2}\left(E[R_A^2] - 2E[R_A]r_f + r_f^2\right) - \left(E[R_A]^2 - 2E[R_A]r_f + r_f^2\right)}} * \sqrt{252}$$
$$= \frac{E[R_A]}{\sqrt{\frac{E[H^2]}{E[H]^2} * E[R_A^2] - E[R_A]^2}} * \sqrt{252} \approx \frac{0.0003721}{\sqrt{\frac{0.509970}{0.26007} * 0.0001443 - 0.00037212}} * \sqrt{252} \approx 0.35$$

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We currently have two teams in operation, Quant Research Team and Macro Research Team. Quant Research Team publishes quantitative finance research reports about incorporating mathematics and statistical methods in finance and portfolio management. Marco Research Team publishes newsletters about ad-hoc global market themes.

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